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# A Necessary and Sufficient Condition for a Product Relation to Be Total

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Let $A, B$, and $C$ be sets, let $\rho$ be a relation on $A \times B$, and let $\sigma$ be a relation on $B \times C$. A necessary and sufficient condition for $\rho \circ \sigma$ to be total is provided in terms of a DeMorgan algebra defined on $B$. 

## 1. INTRODUCTION

Let $A, B$, and $C$ be sets, let $\rho$ be a relation on $A \times B$, and let $\sigma$ be a relation on $B \times C$. The product (composite) relation $\rho \circ \sigma$ on $A \times C$ is the relation

$$\{\langle a, c\rangle \mid \exists\, b \in B\colon a\rho b \text{ and } b\sigma c\}.$$

We are concerned here with determining when $\rho \circ \sigma$ is total, that is, when $\rho \circ \sigma = A \times C$. The product of the relations in Fig. 1a is not total, but the product in Fig. 1b is. Theorem 2 below provides a necessary and sufficient condition for totality in terms of a DeMorgan algebra defined on $B$. The result plays a fundamental role in a combinatorial theory [7] that seeks to extend classical logic to handle dynamic situations.

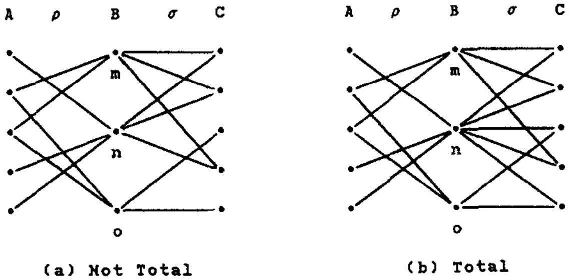


FIG. 1. Product relations.



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We adopt the following notation. $\mathscr{P}(A)$ denotes the power set (set of all subsets) of the set $A$. For a relation $\rho$ on $A \times B$ and for $a \in A$ and $Q \subseteq A$,

$$\rho(a) = \{b \in B \mid a\rho b\}$$
$$\rho(Q) = \{b \in B \mid \exists\, a \in Q\colon a\rho b\}$$

$\rho^{-1}$ denotes the converse of $\rho$. $\min(\mathscr{A})$ denotes the family of minimal sets, with respect to set inclusion, of the family of sets $\mathscr{A}$.

## 2. DeMorgan Algebras

Two algebras are defined below that satisfy most of, but not all, the usual properties of a Boolean algebra. Algebras satisfying this reduced set of properties are called DeMorgan algebras [1–3, 5, 6, 8–11].

Definition. A *DeMorgan algebra* is an algebraic system $\langle D, \vee, \wedge, \bar{}\,\rangle$ where $\vee$ and $\wedge$ are binary operations and $\bar{}$ is a unary operation such that:

1. $\langle D, \vee, \wedge \rangle$ forms a distributive lattice,
2. $\overline{(x \vee y)} = \bar{x} \wedge \bar{y}$ and $\overline{(x \wedge y)} = \bar{x} \vee \bar{y}$ (DeMorgan's Laws), and
3. $\bar{\bar{x}} = x$.

Property 1. *For elements $x$ and $y$ in a DeMorgan algebra, $x \leqslant y \Leftrightarrow \bar{y} \leqslant \bar{x}$.*

A unary operation $\sim$ on a set $S$ such that $\sim\sim s = s$ for all $s \in S$ is called an involution of $S$. The unary operation of a DeMorgan algebra is thus an involution. The following result provides a method for constructing a DeMorgan algebra from an involution defined on an arbitrary set. (See Balbes and Dwinger [1, p. 212].)

Theorem 1. *Let $S$ be a set with an involution $\sim$, and let $\bar{}$ be the unary operation on $\mathscr{P}(S)$ such that for all $X \subseteq S$,*

$$\bar{X} = S - \{\sim s \mid s \in X\}.$$

*Then $\langle \mathscr{P}(S), \cup, \cap, \bar{}\,\rangle$ is a DeMorgan algebra.*

## 3. The $\Psi$ and $\Omega$ Sets

In Theorem 1, let $S = \mathscr{P}(N)$ for an arbitrary set $N$ and let $\sim$ denote set-theoretic complement. Then by Theorem 1, $\langle \mathscr{P}(\mathscr{P}(N)), \cup, \cap, \bar{}\,\rangle$ is a a DeMorgan algebra. We are interested here in a special subalgebra of this DeMorgan algebra.

DEFINITION. Let $N$ be a set. $\Psi(N)$ is then the set of $X \subseteq \mathscr{P}(N)$ such that for all $P, Q \subseteq N$: $(P \in X$ and $P \subseteq Q) \Rightarrow Q \in X$.

In the terminology of Birkhoff [4, pp. 55–61], $\langle \Psi(N), \subseteq \rangle$ is the ring of all $J$-closed subsets (dual semi-ideals) of $\mathscr{P}(N)$. $\langle \Psi(N), \subseteq \rangle$ is isomorphic to the free distributive lattice generated by $|N|$ symbols, with $O$ and $I$ adjoined. To make $\langle \Psi(N), \cup, \cap \rangle$ into a DeMorgan algebra, a unary operation is added that is equivalent—on $\Psi(N)$—to the operation defined in Theorem 1 when $S$ is interpreted as $\mathscr{P}(N)$ and $\sim$ as set-theoretic complement.

PROPERTY 2. *Let $N$ be a set and let $^{-}$ be the unary operation on $\Psi(N)$ such that for all $X \in \Psi(N)$,*

$$\bar{X} = \{Q \subseteq N \mid \forall\, P \in X: P \cap Q \neq \phi\}.$$

*Then $\langle \Psi(N), \cup, \cap, ^{-} \rangle$ is a DeMorgan algebra.*

EXAMPLE. The distributive lattice for $\Psi(\{m, n\})$ is shown in Fig. 2. Moreover,

$$\begin{aligned} \bar{\phi} &= \{\phi, \{m\}, \{n\}, \{m, n\}\} \\ \overline{\{\{m\}, \{m, n\}\}} &= \{\{m\}, \{m, n\}\} \\ \overline{\{\{n\}, \{m, n\}\}} &= \{\{n\}, \{m, n\}\} \\ \overline{\{\{m, n\}\}} &= \{\{m\}, \{n\}, \{m, n\}\}. \end{aligned}$$

In this finite example, it is clear that each element of $\Psi(N)$ is determined by its minimal sets. $\{\{m\}, \{m, n\}\}$, for example, is determined by $\{\{m\}\}$. This property is exploited to produce an algebra that is isomorphic to $\langle \Psi(N), \cup, \cap, ^{-} \rangle$ but whose elements have a more compact representation.

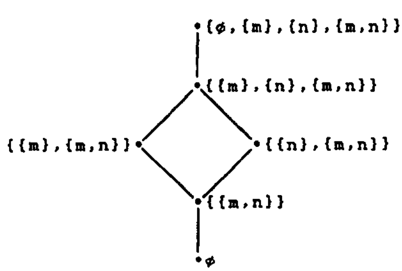


FIG. 2. Distributive lattice for $\Psi(\{m, n\})$.

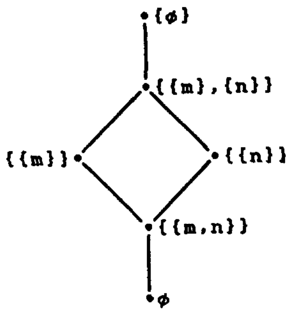


FIG. 3. Distributive lattice for $\Omega(\{m, n\})$.

DEFINITION. Let $N$ be a finite set. $\Omega(N)$ is then the set of $X \subseteq \mathscr{P}(N)$ such that for all $P, Q \in X$: $P \subseteq Q \Rightarrow P = Q$. For $X, Y \in \Omega(N)$,

$$X \vee Y = \min(X \cup Y)$$
$$X \wedge Y = \min(\{P \cup Q \mid P \in X \text{ and } Q \in Y\})$$
$$\bar{X} = \min(\{Q \subseteq N \mid \forall P \in X: P \cap Q \neq \phi\}).$$

PROPERTY 3. *Let $N$ be a finite set. Then* min *is an isomorphism from* $\langle \Psi(N), \cup, \cap, \bar{\ } \rangle$ *to* $\langle \Omega(N), \vee, \wedge, \bar{\ } \rangle$.

PROPERTY 4. *For $X, Y \in \Omega(N)$, $X \leqslant Y$ if and only if for each $P$ in $X$, there exists $Q$ in $Y$ such that $Q \subseteq P$.*

EXAMPLE. The distributive lattice for $\Omega(\{m, n\})$ is shown in Fig. 3.

## 4. Total Relations

The $\Psi$ set defined above is used now in establishing a necessary and sufficient condition for a product relation to be total.

PROPERTY 5. *Let $A, B$, and $C$ be sets, let $\rho$ be a relation on $A \times B$, and let $\sigma$ be a relation on $B \times C$. Then*

$$\{P \subseteq B \mid \rho^{-1}(P) = A\} \quad \textit{and} \quad \{Q \subseteq B \mid \exists\, a \in A: \rho(a) \subseteq Q\}$$
$$\{P \subseteq B \mid \sigma(P) = C\} \quad \textit{and} \quad \{Q \subseteq B \mid \exists\, c \in C: \sigma^{-1}(c) \subseteq Q\}$$

*are elements of $\Psi(B)$.*

LEMMA 1. *Let $A$ and $B$ be sets and let $\rho$ be a relation on $A \times B$. Then*

$$\overline{\{P \subseteq B \mid \rho^{-1}(P) = A\}} = \{Q \subseteq B \mid \exists\, a \in A : \rho(a) \subseteq Q\}.$$

*Proof.* Let

$$X = \{P \subseteq B \mid \rho^{-1}(P) = A\}$$
$$Y = \{Q \subseteq B \mid \exists\, a \in A : \rho(a) \subseteq Q\}.$$

Suppose that there exists $R \in \bar{X}$ such that

$$\forall\, a \in A : \exists\, b \in B: \qquad a \in \rho^{-1}(b) \text{ and } b \notin R.$$

Hence $\rho^{-1}(B - R) = A$ and $(B - R) \in X$. But because $R \in \bar{X}$, $R \cap (B - R) \neq 0$—a contradiction. Thus for all $R \in \bar{X}$,

$$\exists\, a \in A : \forall\, b \in B: \qquad a \in \rho^{-1}(b) \Rightarrow b \in R.$$

Or equivalently, for all $R \in \bar{X}$,

$$\exists\, a \in A: \qquad \rho(a) \subseteq R.$$

In other words, $\bar{X} \subseteq Y$.

From the definition of $Y$, it follows that for all $Q \in Y$, there exists $a \in A$ such that $\rho(a) \subseteq Q$. From the definition of $X$, it follows that for all $P \in X$, for all $a \in A, P \cap \rho(a) \neq \phi$. Hence for all $P \in X$, for all $Q \in Y, P \cap Q \neq \phi$. And so $Y \subseteq \bar{X}$.

THEOREM 2. *Let $A, B$, and $C$ be sets, let $\rho$ be a relation on $A \times B$, and let $\sigma$ be a relation on $B \times C$. Then $\rho \circ \sigma$ is total if and only if*

$$\overline{\{P \subseteq B \mid \rho^{-1}(P) = A\}} \subseteq \{Q \subseteq B \mid \sigma(Q) = C\}.$$

*Proof.*

$$\rho \circ \sigma \text{ is total}$$

if and only if

$$\forall\, a \in A: \qquad \sigma(\rho(a)) = C$$

if and only if

$$\forall\, a \in A : \exists\, Q \subseteq B: \qquad Q \subseteq \rho(a) \text{ and } \sigma(Q) = C$$

if and only if

$$\forall\, P \subseteq B: \qquad (\exists\, a \in A : P = \rho(a)) \Rightarrow (\exists\, Q \subseteq B : \sigma(Q) = C \text{ and } Q \subseteq P)$$

if and only if

$$\forall P \subseteq B: \qquad (\exists a \in A: \rho(a) \subseteq P) \Rightarrow (\exists Q \subseteq B: \sigma(Q) = C \text{ and } Q \subseteq P)$$

if and only if

$$\{P \subseteq B \mid \exists a \in A: \rho(a) \subseteq P\} \subseteq \{Q \subseteq B \mid \sigma(Q) = C\}$$

if and only if (by Lemma 1)

$$\overline{\{P \subseteq B \mid \rho^{-1}(P) = A\}} \subseteq \{Q \subseteq B \mid \sigma(Q) = C\}.$$

In those cases where $B$ is finite, Property 3 permits the condition in Theorem 2 to be replaced by the corresponding condition involving elements from $\Omega(B)$.

COROLLARY 1. *Let $A, B$, and $C$ be sets with $B$ finite, let $\rho$ be a relation on $A \times B$, and let $\sigma$ be a relation on $B \times C$. Then $\rho \circ \sigma$ is total if and only if*

$$\overline{\min(\{P \subseteq B \mid \rho^{-1}(P) = A\})} \leqslant \min(\{Q \subseteq B \mid \sigma(Q) = C\}).$$

EXAMPLE. Corollary 1 and Property 4 are applied to the two situations in Fig. 1. For the relations in Fig. 1a,

$$\min(\{P \subseteq B \mid \rho^{-1}(P) = A\}) = \{\{m, n\}, \{n, o\}\}$$
$$\overline{\min(\{P \subseteq B \mid \rho^{-1}(P) = A\})} = \{\{n\}, \{m, o\}\}$$
$$\min(\{P \subseteq B \mid \sigma(P) = C\}) = \{\{m, o\}, \{n, o\}\}.$$

In this case,

$$\overline{\min(\{P \subseteq B \mid \rho^{-1}(P) = A\})} \nleqslant \min(\{Q \subseteq B \mid \sigma(Q) = C\})$$

and $\rho \circ \sigma$ is not total. For the relations in Fig. 1b,

$$\min(\{P \subseteq B \mid \rho^{-1}(P) = A\}) = \{\{m, n\}, \{n, o\}\}$$
$$\overline{\min(\{P \subseteq B \mid \rho^{-1}(P) = A\})} = \{\{n\}, \{m, o\}\}$$
$$\min(\{P \subseteq B \mid \sigma(P) = C\}) = \{\{n\}, \{m, o\}\}.$$

Here,

$$\overline{\min(\{P \subseteq B \mid \rho^{-1}(P) = A\})} \leqslant \min(\{Q \subseteq B \mid \sigma(Q) = C\})$$

and $\rho \circ \sigma$ is total.

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